
Conformal Field Theory and Gravity

Solutions to Problem Set 7

Fall 2024

1. Spinning fields

- (a) Let's go through the Ward identities for translations, dilatations, rotations, and special conformal transformations.

Translational invariance requires

$$\langle j_\mu(x) j_\nu(y) \rangle = f_{\mu\nu}(x - y)$$

for some function $f_{\mu\nu}$ that depends only on $x - y$.

Rotational covariance requires $f_{\mu\nu}(x - y)$ to transform as a rank-2 tensor under rotations. The only independent symmetric rank-2 tensor structures available are $\delta_{\mu\nu}$ and $\frac{(x-y)_\mu(x-y)_\nu}{|x-y|^2}$. Thus, we can write the general form of $f_{\mu\nu}(x - y)$ as:

$$f_{\mu\nu}(x - y) = g(x - y) \left(\delta_{\mu\nu} + B \frac{(x - y)_\mu(x - y)_\nu}{|x - y|^2} \right)$$

where B is a constant to be determined, and g is an undetermined function. Dilatations with scaling dimension Δ require a power-law behaviour for g :

$$g(x - y) = \frac{1}{|x - y|^{2\Delta}} \quad (1)$$

Therefore

$$f_{\mu\nu}(x - y) = \frac{1}{|x - y|^{2\Delta}} \left(\delta_{\mu\nu} + B \frac{(x - y)_\mu(x - y)_\nu}{|x - y|^2} \right)$$

- (b) We will derive the Ward identity for the special conformal transformation, as the other equations are very similar.

Since $\xi^\mu(x) \propto x$ and we set $y = 0$, we only need to consider the transformation of $j_\mu(x)$. Using the following

$$\begin{aligned} \frac{\partial x'^\nu}{\partial x'^\mu} &= \delta_\mu^\nu + 2(b \cdot x) \delta_\mu^\nu + 2(b_\mu x^\nu - b^\nu x_\mu) \\ j_\mu(x') &= j_\mu(x) + (2(b \cdot x) x^\nu - x^2 b^\nu) \partial_\nu j_\mu(x) \end{aligned} \quad (2)$$

we can expand the tensor transformation formula to first order, and find $j'_\mu(x) = j_\mu(x) + \delta j_\mu(x)$, with

$$\delta j_\mu = (2\Delta(b \cdot x) \delta_\mu^\nu + 2(b_\mu x^\nu - b^\nu x_\mu)) j_\nu + (2(b \cdot x) x^\nu - x^2 b^\nu) \partial_\nu j_\mu(x) \quad (3)$$

Plugging this into the Ward identity, we get the desired result

$$\begin{aligned} \langle \delta j_\mu(x) j_\nu(0) \rangle &= 0 \\ \left\{ (2\Delta(b \cdot x) + 2(b \cdot x) x^\rho \partial_\rho + x^2 b^\rho \partial_\rho) \delta_\lambda^\mu + 2(b_\lambda x^\mu - b^\mu x_\lambda) \right\} \langle j_\mu(x) j_\nu(0) \rangle &= 0 \end{aligned} \quad (4)$$

- (c) The first two properties from the first hint are immediate to check, so let us verify the third one.

From the first property we obtain that $\det(I_\mu^\rho) = \pm 1$, therefore

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{|\det(I_\mu^\rho)|}{x^{2d}} = \frac{1}{x^{2d}} \quad (5)$$

Now, using the vector transformation under inversion we find

$$j'_\mu(x) = \left| \frac{\partial x'}{\partial x} \right|^{\frac{\Delta-1}{d}} \frac{I_\mu^\nu(x)}{x^2} j_\nu(x') = \frac{I_\mu^\nu(x)}{x^{2\Delta}} j_\nu(x') \quad (6)$$

Hence we need to check

$$\langle j'_\mu(x) j'_\nu(y) \rangle = \frac{I_\mu^\rho(x)}{x^{2\Delta}} \frac{I_\nu^\sigma(y)}{y^{2\Delta}} \langle j_\rho(x') j_\sigma(y') \rangle \doteq \langle j_\mu(x) j_\nu(y) \rangle \quad (7)$$

Equivalently,

$$\frac{I_\mu^\rho(x)}{x^{2\Delta}} \frac{I_\nu^\sigma(y)}{y^{2\Delta}} f_{\rho\sigma}(x' - y') \doteq f_{\mu\nu}(x - y) \quad (8)$$

Let us expand the LHS, using $y \rightarrow 0, y \rightarrow \infty$, and use the fact that $\frac{1}{|x' - y'|^{2\Delta}} = \frac{x^{2\Delta} y^{2\Delta}}{|x - y|^{2\Delta}}$

$$\begin{aligned} \frac{I_\mu^\rho(x)}{x^{2\Delta}} \frac{I_\nu^\sigma(y)}{y^{2\Delta}} f_{\rho\sigma}(x' - y') &= \frac{I_\mu^\rho(x)}{x^{2\Delta}} \frac{I_\nu^\sigma(y)}{y^{2\Delta}} \frac{I_{\rho\sigma}(x' - y')}{|x' - y'|^{2\Delta}} \\ &= I_\mu^\rho(x) I_\nu^\sigma(y) \frac{I_{\rho\sigma}(y')}{|x - y|^{2\Delta}} \left(1 + \mathcal{O}(x'/y') \right) \\ &= I_\mu^\rho(x) I_\nu^\sigma(y) \frac{I_{\rho\sigma}(y)}{|x - y|^{2\Delta}} \left(1 + \mathcal{O}(x'/y') \right) \\ &= \frac{I_{\mu\nu}(x)}{|x - y|^{2\Delta}} \left(1 + \mathcal{O}(x'/y') \right) \\ &= \frac{I_{\mu\nu}(x - y)}{|x - y|^{2\Delta}} \left(1 + \mathcal{O}(x'/y') \right) \\ &\lim_{y \rightarrow 0} f_{\mu\nu}(x - y) \end{aligned} \quad (9)$$

Note that this derivation would have proceeded similarly even if you tried to use a general value of y .

- (d) Consider $y = 0$. Then

$$\partial_\mu \langle j^\mu(x) j^\nu(0) \rangle \propto -2\Delta x_\mu I^{\mu\nu}(x) + x^2 \partial_\mu I^{\mu\nu} \propto (\Delta - d + 1) = 0 \quad (10)$$

Hence $\Delta = d - 1$. This ensures that the conserved charge $Q \equiv \int d^{d-1}x \, n^\mu j_\mu$ is scale-invariant.

- (e) The proof is similar to point 1 and 2, but more tedious. The efficient way to do this is via the embedding space formalism.

Analogously to point 3, we require

$$\partial_\mu \langle T^{\mu\nu}(x) T^{\rho\sigma}(0) \rangle \propto (\Delta - d) = 0 \quad (11)$$

Hence $\Delta = d - 1$. This ensures that the charges generating conformal transformations are scale-invariant.

2. Introduction to the embedding space formalism

(a) Using that

$$dP_0^2 = dP_{d+1}^2 = x_\mu x_\nu dx^\mu dx^\nu \quad dP^\mu = dx^\mu \quad (12)$$

it is straightforward to see that

$$ds^2 = -dP_0^2 + dP_{d+1}^2 + (dP^\mu)^2 = \delta_{\mu\nu} dx^\mu dx^\nu \quad (13)$$

(b) Let us take

$$P^0 = \Omega(x) \frac{1+x^2}{2} \quad P^{d+1} = \Omega(x) \frac{1-x^2}{2} \quad P^\mu = \Omega(x) x^\mu \quad (14)$$

Then,

$$dP^0 = (\partial_\mu \Omega) \frac{1+x^2}{2} dx^\mu + \Omega x_\mu dx^\mu \quad (15)$$

$$dP^{d+1} = (\partial_\mu \Omega) \frac{1-x^2}{2} dx^\mu - \Omega x_\mu dx^\mu \quad (16)$$

$$dP^\mu = (\partial_\nu \Omega) x^\mu dx^\nu + \Omega dx^\mu \quad (17)$$

Then by expanding explicitly all terms, we find that

$$ds^2 = -dP_0^2 + dP_{d+1}^2 + (dP^\mu)^2 = \Omega^2 \delta_{\mu\nu} dx^\mu dx^\nu \quad (18)$$

(c) Let us start by the two-point function. By Lorentz invariance, $\langle \mathcal{O}(P_1) \mathcal{O}(P_2) \rangle$ can only be a function of scalar invariants built from P_1 and P_2 , i.e. P_1^2 , P_2^2 and $P_1 \cdot P_2$. Since we are on the light-cone, $P_1^2 = P_2^2 = 0$, thus

$$\langle \mathcal{O}(P_1) \mathcal{O}(P_2) \rangle = f(P_1 \cdot P_2) \quad (19)$$

Since this needs to satisfy the homogeneity property, it can only be a power-law,

$$f(P_1 \cdot P_2) = \text{const} |P_1 \cdot P_2|^\alpha \quad (20)$$

where α is determined by the homogeneity property

$$f(\lambda P_1 \cdot P_2) = \lambda^{-2\Delta} f(P_1 \cdot P_2) \implies \alpha = -\Delta \quad (21)$$

Thus

$$\langle \mathcal{O}(P_1) \mathcal{O}(P_2) \rangle = \frac{\text{const}}{|P_1 \cdot P_2|^\Delta} \quad (22)$$

Note that on the physical section,

$$P_1(x_1) \cdot P_2(x_2) = - \left(\frac{1+x_1^2}{2} \frac{1+x_2^2}{2} \right) + \left(\frac{1-x_1^2}{2} \frac{1-x_2^2}{2} \right) + x_1 \cdot x_2 = -\frac{1}{2} (x_1 - x_2)^2 \quad (23)$$

Thus,

$$\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \rangle = \frac{\text{const}'}{|x_1 - x_2|^{2\Delta}} \quad (24)$$

For the 3-point function, we have

$$\langle \mathcal{O}(P_1) \mathcal{O}(P_2) \mathcal{O}(P_3) \rangle = f(P_1 \cdot P_2, P_1 \cdot P_3, P_2 \cdot P_3) \quad (25)$$

Note that the correlator is invariant under permutations of $(1, 2, 3)$, thus f should be invariant under permutations of its 3 arguments. Also, the homogeneity property implies again a power-law. Thus,

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3) \rangle = (|P_1 \cdot P_2||P_1 \cdot P_3||P_2 \cdot P_3|)^\alpha \quad (26)$$

Finally, α is again determined from scaling which implies

$$\lambda^{6\alpha} = \lambda^{-3\Delta} \implies \alpha = -\frac{\Delta}{2} \quad (27)$$

Thus,

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3) \rangle = \frac{\text{const}}{|P_1 \cdot P_2|^{\Delta/2}|P_1 \cdot P_3|^{\Delta/2}|P_2 \cdot P_3|^{\Delta/2}} \quad (28)$$

On the physical section, using $P_1(x_1) \cdot P_2(x_2) \propto (x_1 - x_2)^2$ as previously, we obtain

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3) \rangle = \frac{\text{const}'}{|x_{12}|^\Delta|x_{13}|^\Delta|x_{23}|^\Delta} \quad (29)$$

- (d) For the two-point function of vector operators $\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2) \rangle$, we impose Lorentz covariance by noting that the result must be built out of $P_1^A P_2^B$ or η^{AB} , with Lorentz contractions that can only be made of $P_1 \cdot P_2$, namely

$$\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2) \rangle = a_1\eta^{AB}f_1(P_1 \cdot P_2) + a_2P_1^B P_2^A f_2(P_1 \cdot P_2) + a_3P_1^A P_2^B f_3(P_1 \cdot P_2) \quad (30)$$

where f_1 , f_2 and f_3 are power-laws because of homogeneity. Thus,

$$\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2) \rangle = a_1\eta^{AB}|P_1 \cdot P_2|^{\alpha_1} + a_2P_1^B P_2^A |P_1 \cdot P_2|^{\alpha_2} + a_3P_1^A P_2^B |P_1 \cdot P_2|^{\alpha_3} \quad (31)$$

Homogeneity implies

$$\alpha_1 = -\Delta \quad \alpha_2 = \alpha_3 = -(\Delta + 1) \quad (32)$$

Thus,

$$\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2) \rangle = \frac{a'_1(P_1 \cdot P_2)\eta^{AB} + a'_2P_2^A P_1^B + a'_3P_1^A P_2^B}{(-2P_1 \cdot P_2)^{\Delta+1}} \quad (33)$$

We also impose that

$$(P_1)_A \langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2) \rangle = (P_2)_B \langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2) \rangle = 0 \quad (34)$$

which implies $a'_2 = -a'_1$. Thus,

$$\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2) \rangle = \text{const.} \frac{(P_1 \cdot P_2)\eta^{AB} - P_2^A P_1^B}{(-2P_1 \cdot P_2)^{\Delta+1}} + a'_3 \frac{P_1^A P_2^B}{(-2P_1 \cdot P_2)^{\Delta+1}} \quad (35)$$

The last term is irrelevant since vector operators are identified by the redundancy $\mathcal{O}^A(P) \sim \mathcal{O}^A(P) + P^A \lambda(P)$.

- (e) To go to physical space,

$$\langle \mathcal{O}_\mu(x_1)\mathcal{O}_\nu(x_2) \rangle = \frac{\partial P_1^A(x_1)}{\partial x_1^\mu} \frac{\partial P_2^B(x_2)}{\partial x_1^\nu} \langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2) \rangle \quad (36)$$

Noting that

$$\frac{\partial P_1^A(x_1)}{\partial x_1^\mu} \frac{\partial P_2^A(x_2)}{\partial x_2^\nu} \eta_{AB} = \eta_{\mu\nu} \quad (37)$$

$$\frac{\partial P_1^A(x_1)}{\partial x_1^\mu} P_{2,A}(x_2) = -(x_{1,\mu} - x_{2,\mu}) \quad (38)$$

$$P_1 \cdot P_2 = -\frac{1}{2}|x_1 - x_2|^2 \quad (39)$$

This gives

$$\frac{\partial P_1^A(x_1)}{\partial x_1^\mu} \frac{\partial P_2^B(x_2)}{\partial x_1^\nu} \left(\eta_{AB} - \frac{(P_2)_A (P_1)_B}{(P_1 \cdot P_2)} \right) = I_{\mu\nu}(x_1 - x_2) \quad (40)$$

Thus,

$$\langle \mathcal{O}_\mu(x_1) \mathcal{O}_\nu(x_2) \rangle = \text{const}' \frac{I_{\mu\nu}(x_1 - x_2)}{|x - y|^{2\Delta}} \quad (41)$$

as derived in exercise 1.